LETTER TO THE EDITOR

Maslov indices and monodromy

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Abstract
We prove that for a Hamiltonian system on a cotangent bundle that is Liouville-integrable and has monodromy the vector of Maslov indices is an eigenvector of the monodromy matrix with eigenvalue 1. As a corollary, the resulting restrictions on the monodromy matrix are derived.

PACS numbers: 02.30.Ik, 02.40.Ma, 03.65.Vf, 45.20.Jj
Mathematics Subject Classification: 37J35, 81S10, 53D12

1. Introduction

The Liouville–Arnold theorem describes the local structure of an integrable system: for regular values of the energy–momentum map $F : T^*M \to \mathbb{R}^n$, the preimage of a regular value is an n-torus (or a union of disconnected n-tori, but for simplicity we assume there is just one), and there exist action-angle variables in a neighbourhood of this torus. Thus, locally phase space has the structure of a trivial n-torus-bundle over an open neighbourhood of a regular value in the image of F. Duistermaat [7] pointed out that globally the torus-bundle over the regular values of F may be nontrivial. This phenomenon is called monodromy. As a result, there may not exist global action-angle variables. In 2 degrees of freedom monodromy is well understood [13, 19]. It is a common phenomenon because it occurs in a neighbourhood of an equilibrium of focus–focus type. In 3 degrees of freedom now also many examples [17, 18, 8, 3] are known.

Quantization of a classical system with monodromy leads to quantum monodromy [11, 16, 5, 14, 9]. The fact that the classical actions cannot be globally defined implies that the quantum numbers suffer the same problem.

The Maslov index is not only interesting for semiclassical quantization, but also in classical mechanics it is an invariant object defined for paths on Lagrangian submanifolds,

e.g. on invariant tori, see [1, 12, 2]. Recently, it has been shown that the Maslov index is related to the singular points of the energy–momentum map [10].

In this letter, we are going to show that if the vector of Maslov indices is non-zero, then it is an eigenvector of the monodromy matrix with eigenvalue 1. This has some interesting consequences for the structure of admissible monodromy matrices. Since the Maslov index is only defined on cotangent bundles our results are only valid when the phase space is a symplectic manifold of the form T^*M.

2. Maslov indices

Let C be a closed curve in the set of regular values of the energy–momentum map. We take C to be parametrized by $0 \leqslant s \leqslant 1$. Let T_s denote the corresponding one-parameter family of n-tori in phase space. Fix a basis of cycles γ_0 for T_0. By continuation this defines a basis of cycles γ_s for every s. The curve C has monodromy when $\gamma_1 = \mathbf{M}\gamma_0$ for $\mathbf{M} \in SL(n, \mathbb{Z})$ is nontrivial. More precisely, monodromy is a nontrivial automorphism of the first homology group, and it implies that the preimage of C under F is a nontrivial n-torus-bundle over C. The basis γ_s determines actions I_s and Maslov indices μ_s on T_s. In fact, the Maslov indices are independent of s, as they depend continuously on s and are integer-valued [15]. Let us denote their common value by μ. Our main result is the following simple observation:

Theorem 1. *If the vector of Maslov indices μ is not equal to zero, then μ is an eigenvector of the monodromy matrix $\mathbf{M}$ with eigenvalue 1.*

Proof. We have that $\mu_1 = \mathbf{M}\mu_0$ (just as $I_1 = \mathbf{M}I_0$), since in general a change of basis cycles $\gamma' = \mathbf{T}\gamma$, where $\mathbf{T} \in SL(n, \mathbb{Z})$, induces the transformation of Maslov indices $\mu' = \mathbf{T}\mu$ (and the transformation of actions $I' = \mathbf{T}I$). Since $\mu_s = \mu$ for all s, $\mu_1 = \mathbf{M}\mu_0$, i.e.

$$\mathbf{M}\mu = \mu.$$ □

We remark that the Maslov indices μ, the actions I and the monodromy matrix M depend on the initial choice of basis γ_0. Under a change of basis $\gamma_0' = \mathbf{T}\gamma_0$, where $T \in SL(n, \mathbb{Z})$, we have that $\mu' = \mathbf{T}\mu$ and $\mathbf{M}' = \mathbf{TMT}^{-1}$.

3. Monodromy matrices

From theorem 1, we immediately obtain the well-known result [13, 19] about the structure of monodromy matrices in 2 degrees of freedom:

Corollary 2. *For $n = 2$ degrees of freedom and a loop C with $\mu \neq 0$ there exists a basis of cycles such that the monodromy matrix of C has the form*

$$\mathbf{M} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}.$$

Proof. Since $\mathbf{M} \in SL(2, \mathbb{Z})$ the eigenvalues λ_1, λ_2 must satisfy $\lambda_1\lambda_2 = 1$. But one eigenvalue must be 1 by theorem 1, hence $\lambda_1 = \lambda_2 = 1$. Finally, a matrix in $SL(2, \mathbb{Z})$ with a single eigenvalue equal to 1 is conjugate to the stated form by some matrix from $SL(2, \mathbb{Z})$. The Maslov index in this basis is $\mu = (\mu_1, 0)$.   □

Note that this does not give a complete classification of monodromy matrices on cotangent bundles because we have assumed that $\mu \neq 0$. When $\mu \neq 0$, corollary 2 is quite strong because

no assumption is needed on the type of singularity that is encircled by C, in particular the usual non-degeneracy condition is not needed.

Corollary 2 is a special case of the simple general

Lemma 3. *Suppose* $\mathbf{M} \in SL(n, \mathbb{Z})$ *has eigenvalue* ± 1. *Then there exists* $\mathbf{T} \in SL(n, \mathbb{Z})$ *such that* $\mathbf{M}' = \mathbf{T}\mathbf{M}\mathbf{T}^{-1}$ *has first column equal to* $\pm \mathbf{e}_1 = (\pm 1, 0, \ldots, 0)^t$.

Proof. Let $\mathbf{u}$ denote an eigenvector of $\mathbf{M}$ with eigenvalue ± 1, chosen so that its components are coprime integers. Then one can construct a matrix $\mathbf{S} \in SL(n, \mathbb{Z})$ whose first column is $\mathbf{u}$ (see, e.g., [4]). Let $\mathbf{T} = \mathbf{S}^{-1}$ and $\mathbf{M}' = \mathbf{T}\mathbf{M}\mathbf{T}^{-1}$. It is easy to check that $\mathbf{e}_1$ is an eigenvector of $\mathbf{M}'$ with eigenvalue ± 1, so that $\mathbf{M}'$ has first column equal to $\pm \mathbf{e}_1$.   □

Using lemma 3 and again the fact that $\det \mathbf{M} = 1$ and $\lambda_1 = 1$, we can obtain the classification of monodromy matrices (for non-zero Maslov index) in $n = 3$ degrees of freedom:

Corollary 4. *For $n = 3$ degrees of freedom and a loop C with $\mu \neq 0$ there exists a basis of cycles such that the monodromy matrix $\mathbf{M}$ of C has one of the following forms:*

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & * & * \\ 0 & -1 & * \\ 0 & 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & * \\ 0 & \mathbf{B} \end{pmatrix}, \tag{1}$$

where $\mathbf{B} \in SL(2, \mathbb{Z})$ *has irrational eigenvalues and * denotes integers.*

Proof. The eigenvalue 1 can appear with algebraic multiplicity $m_a = 1$ or $m_a = 3$ only; $m_a = 2$ is impossible because $\det \mathbf{M} = \lambda_1 \lambda_2 \lambda_3 = 1$.[4] The case $m_a = 3$ corresponds to the first form above. When $m_a = 1$, the remaining eigenvalues are either both -1, corresponding to the second form, or they are irrational, corresponding to the third. Other combinations of eigenvalues are not possible, because rational eigenvalues of matrices in $SL(n, \mathbb{Z})$ are necessarily equal to ± 1.

If the eigenvalues are all ± 1 (corresponding to the first two forms), the matrices can be made upper triangular by applying lemma 3 recursively, using a transformation of the form

$$\mathbf{T}_n = \begin{pmatrix} 1 & * \\ \mathbf{0} & \mathbf{T}_{n-1} \end{pmatrix}.$$

Matrices with two irrational eigenvalues cannot be made upper triangular in $SL(n, \mathbb{Z})$ (as the diagonal elements of a triangular matrix are its eigenvalues).   □

It is interesting to consider how the entries denoted * in (1) can be normalized. In the first form, the eigenvalue 1 has geometric multiplicity m_g equal to 1 or 2 (i.e., there are either one or two independent eigenvectors with eigenvalue 1). The normal form for $m_g = 2$ has been computed in [17]. The result is that only a single non-zero element remains above the diagonal. Essentially this means that when $m_g = 2$, the matrix can be block-diagonalized in $SL(3, \mathbb{Z})$. In the remaining cases in (1) a block-diagonal form is in general not possible: conjugating a block triangular matrix with a block triangular matrix gives

$$\begin{pmatrix} 1 & -\mathbf{d}\mathbf{D}^{-1} \\ \mathbf{0} & \mathbf{D}^{-1} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{a} \\ \mathbf{0} & \mathbf{A} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{d} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} = \begin{pmatrix} 1 & (\mathbf{a} - \mathbf{d}\mathbf{D}^{-1}(\mathbf{A} - 1))\mathbf{D} \\ \mathbf{0} & \mathbf{D}^{-1}\mathbf{A}\mathbf{D} \end{pmatrix}.$$

Setting the upper right element of the right-hand side to zero and solving for $\mathbf{d}$ involves the inverse of $\mathbf{A} - \mathbf{1}$ which is in general not an integer matrix. Using a more general transformation

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[4] For general n the multiplicity cannot be $n - 1$.

leads to the same condition. Thus for general $\mathbf{a}$ and $\mathbf{A}$, the monodromy matrix cannot be block-diagonalized. However, e.g., for the special matrix $\mathbf{A} = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix} \right)$ (the 'cat map'), it is always possible since $\det(\mathbf{A} - \mathbf{1}) = -1$. If $\mathbf{A} - \mathbf{1}$ is singular (corresponding to the first case in corollary 4), the resulting Diophantine equation may or may not have a solution.

Our results need the condition $\mu \neq 0$. It may be possible to show that $\mu \neq 0$ necessarily holds for certain configuration spaces M. We suspect, for example, that this is the case when $M = \mathbb{R}^n$, although we have not been able to prove this. Our result would then give the complete classification of monodromy matrices on $T^*\mathbb{R}^n$. In particular, the construction of arbitrary monodromy matrices given in [6] would be impossible on these cotangent bundles.

In 3 degrees of freedom, the known examples of monodromy are either of the first form with $m_g = 2$ [17, 18] or of the last form and block-diagonal. The last form of $\mathbf{M}$ is realized for geodesic flows on Sol-manifolds, where an arbitrary hyperbolic $\mathbf{B} \in SL(2, \mathbb{Z})$ may appear [3].

The main implication of the above is that when $m_a = 3$ and $m_g = 2$ there are always two invariant actions, i.e. actions that do not change globally along the path C. Obviously, there is always one invariant action, namely the one corresponding to the eigenvector $\mathbf{e}_1$, and when $m_g = 1$ it is the only one. With eigenvalues -1 there is at most one invariant action, but another action is invariant when C is traversed twice. Hence, on a covering space this may reduce to $m_a = 3$ and $m_g = 2$. It would be very interesting to find an example of this type.

Acknowledgments

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